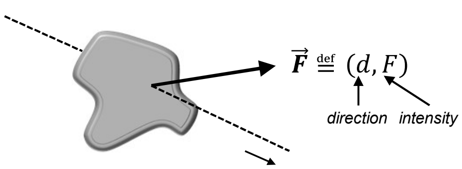
UCONN
UNIVERSITY OF CONNECTICUT

Introduction to Biomechanics Part II

ENGR 1166 Biomedical Engineering

Recap

A **force** is an interaction that tends to change the motion of an object with mass



Recap

A **force** is an interaction that tends to change the motion of an object with mass

A change in net force applied on an object with mass produces an **acceleration** in the direction of the force with a magnitude in proportion to the force (**mass**)

$$\sum_i \vec{F}_i = m\vec{a}$$

How do we measure?

International System of Units
(mass based system)

Meter (m)	*length*
Second (s)	*time*
Kilogram (kg)	*mass*

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Newton (N) *force*

1 N $\stackrel{\text{def}}{=}$ Force required to give a 1 kg mass an acceleration of 1 m/s^2

How do we measure?

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U.S. Customary Units (force based system)

length	Foot (ft)
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1 slug $\stackrel{\text{def}}{=}$ mass given an acceleration of 1 ft/s^2 when a force of 1 lbf is applied

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$$F = ma$$

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$$F = ma = (mg)\frac{a}{g} = w\frac{a}{g}$$

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↓

mass Slug (slug)

$$F = ma = (mg)\frac{a}{g} = w\frac{a}{g}$$

$$\boldsymbol{slug} = {}^{\boldsymbol{lbf}}/_{\boldsymbol{g}} \qquad (g = 32.2 \text{ ft/s}^2)$$

Equivalency of the systems

International System of Units (mass based system)

$$[N] = [kg]\frac{[m]}{[s]^2}$$

U.S. Customary Units (force based system)

$$[lbf] = [slug]\frac{[ft]}{[s]^2}$$

$$10\ N = 10\ kg\frac{m}{s^2} = 10\ \frac{2.2}{32.2}\ slug\ \frac{3.28\ ft}{s^2} = 2.24\ lbf$$

Dimensional homogeneity

In order to check correctness of calculations, let us use:

$F \stackrel{\text{def}}{=}$ force $\quad M \stackrel{\text{def}}{=}$ mass $\quad L \stackrel{\text{def}}{=}$ length $\quad T \stackrel{\text{def}}{=}$ time

and let us express other quantities in terms of F, M, L, T:

$$velocity \stackrel{\text{def}}{=} \frac{length}{time} \equiv L/T$$

$$acceleration \stackrel{\text{def}}{=} length/time \cdot time \equiv L/T^2$$

Then, to check the correctness of our equation we look at the dimensions on each side of the equation to see if they match.

Dimensional homogeneity

For example:

$$Fx = \frac{1}{2}m \cdot v^2$$

F =force; x =length; m =mass; v =velocity

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$$[F][L] = [M]([L]/[T])^2$$

$m = F/a$

$$[F][L] = \frac{[F]}{[L]/[T]^2}([L]/[T])^2$$

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NOTE: *This shows correctness in terms of dimensions. We also need to check the equation in terms of the units used!*

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$$[N][m]=[N][m]$$

Common Errors:

- *Different systems* $[kg][m]^2/[s]^2 \neq [kg]\mathbf{[ft]}^2/[s]^2$
- *Mixed units* $[kg][m]^2/[s]^2 \neq [kg]\mathbf{[cm]}^2/[s]^2$

Numerical precision and accuracy

- Dividing and multiplying numbers

  Rule of Thumb: *When dividing or multiplying two numbers the results is only accurate to the same number of digits as the least accurate number.*

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- Rounding Numbers

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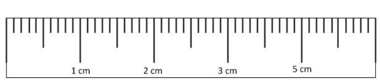
  0.45128 rounds to 0.4513 or 0.451 or 0.45 or 0.5

  depending on the precision of the numbers used

Numerical precision and accuracy

- Taking a measurement

What is the smallest value that we can record from the scale?



1 millimeter (mm)

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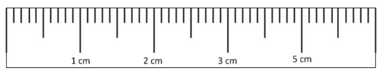

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While a measurement might fall between two mm markings on the scale we cannot be more accurate then ±0.5 mm when we record our data.

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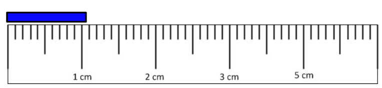

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While a measurement might fall between two mm markings on the scale we cannot be more accurate then ±0.5 mm when we record our data.

If we were to measure the blue rectangle with the scale the value would be recorded as **10 mm ± 0.5 mm**

Vectors

- A vector is a quantity that has magnitude and direction

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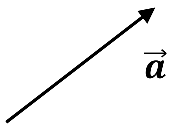
- A vector is drawn as an arrow



$\vec{a}$

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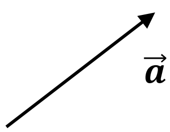

$\vec{a}$

- **Length** → *proportional to the magnitude*
- **Position*** → *direction of the vector*

** in a polar system*

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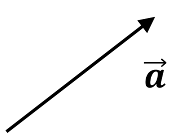

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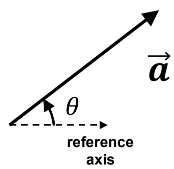

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- **Length** → *proportional to the magnitude*
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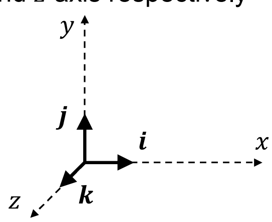
Unit vectors

- Have a unit length (1) and are defined along the x, y and z axis
- Notation $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ represent the unit vectors in the x, y and z axis respectively



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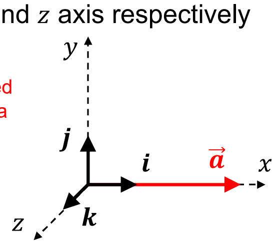
A vector can be represented by using a unit vector and a scalar, e.g.:

$\vec{a} = 10\boldsymbol{i}$

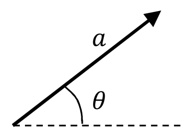


Equivalent representations

$\vec{\boldsymbol{a}} = (\theta, a)$

direction
position (angle) in a polar system

magnitude
scalar



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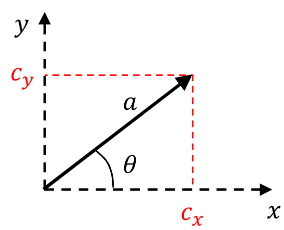

Vector $\vec{\boldsymbol{a}}$ is composed of unit vector $\boldsymbol{i}$ and scalar c_x in the x-direction and unit vector $\boldsymbol{j}$ and scalar c_y in the y- direction:

$$\vec{\boldsymbol{a}} = c_x\boldsymbol{i} + c_y\boldsymbol{j}$$

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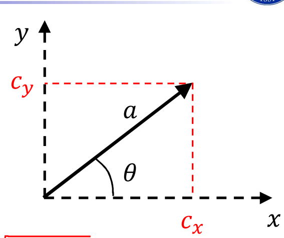


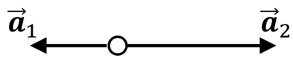
$$a = \sqrt{c_x^2 + c_y^2}$$

$$\theta = \arctan\left(\frac{c_y}{c_x}\right)$$

Vectors: definitions

- Collinear

Vectors pass through or lie on the same line of action



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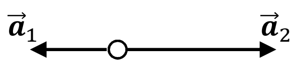
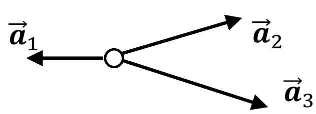


- Concurrent

Vectors have a common point of intersection



Sum of vectors

- Collinear vectors

$$\vec{a}_r = \vec{a}_1 + \vec{a}_2$$

Sum of vectors

- Collinear vectors

$\vec{\boldsymbol{a}}_1$ $\vec{\boldsymbol{a}}_2$ $\vec{\boldsymbol{a}}_r$

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$\vec{\boldsymbol{a}}_1 = (180°, 4)$
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$\vec{\boldsymbol{a}}_1 = (180°, 4)$
$\vec{\boldsymbol{a}}_2 = (0°, 10)$

$\vec{\boldsymbol{a}}_r = (0°, 6)$

Sum of vectors

- Collinear vectors

- Concurrent vectors

Sum of vectors

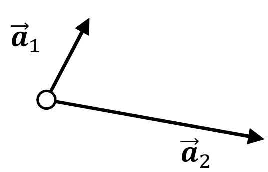
- Collinear vectors

- Concurrent vectors
  - Trigonometric
  - Direct measurement
  - Vector addition

parallelogram law

Sum of vectors

- Trigonometric (parallelogram law)



Sum of vectors

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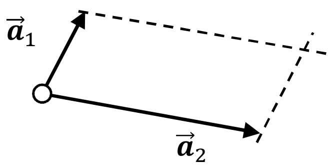

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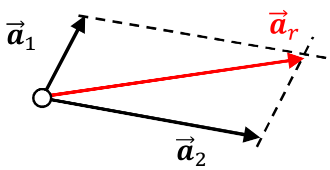

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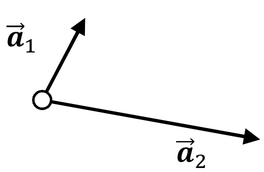

- Direct measurement (head to tail rule)

Sum of vectors

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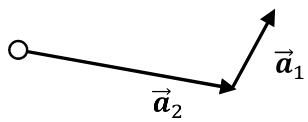

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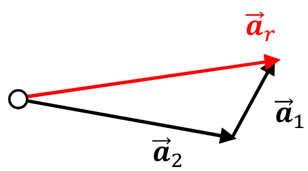

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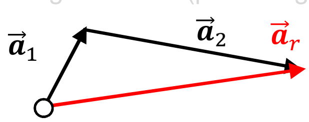
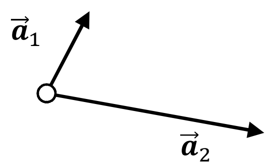


- Direct measurement (head to tail rule)

Sum of vectors

- Vector addition



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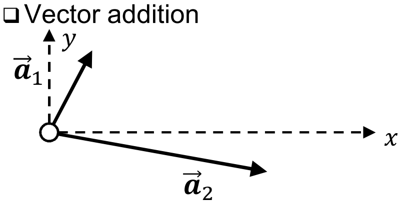

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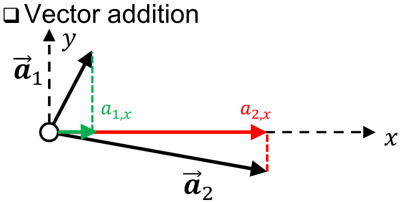

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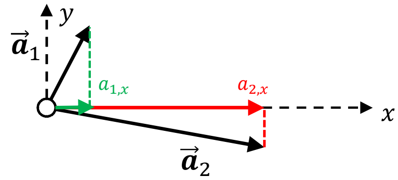
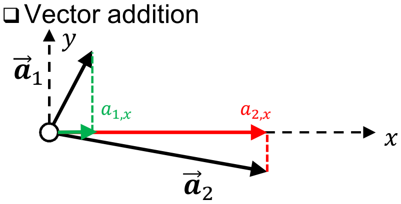


$\vec{a}_1 = a_{1,x}\boldsymbol{i} + a_{1,y}\boldsymbol{j}$ $\qquad$ $\vec{a}_2 = a_{2,x}\boldsymbol{i} + a_{2,y}\boldsymbol{j}$

Sum of vectors

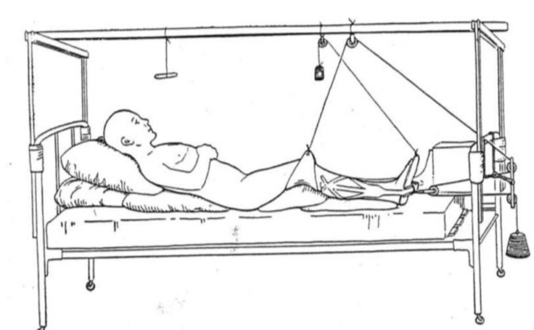
- Vector addition



$\vec{a}_1 = a_{1,x}\boldsymbol{i} + a_{1,y}\boldsymbol{j}$ $\qquad$ $\vec{a}_2 = a_{2,x}\boldsymbol{i} + a_{2,y}\boldsymbol{j}$

$$\vec{a}_1 + \vec{a}_2 = (a_{1,x} + a_{2,x})\boldsymbol{i} + (a_{1,y} + a_{2,y})\boldsymbol{j} = \vec{a}_r$$

An example: Russell's traction



Source: Am. J. Nursing, Vol. 43(5) May 1943, Page 464.

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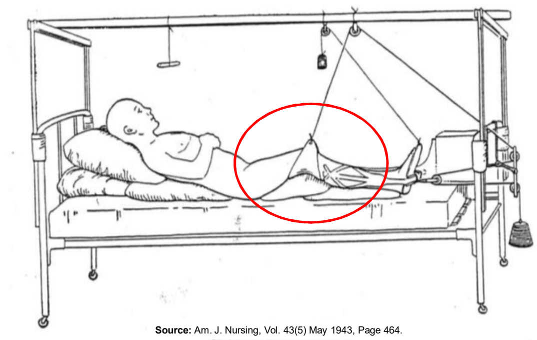

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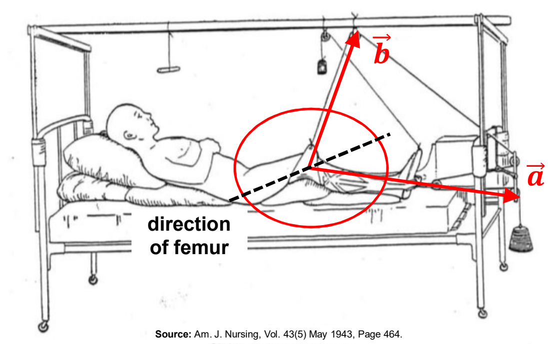

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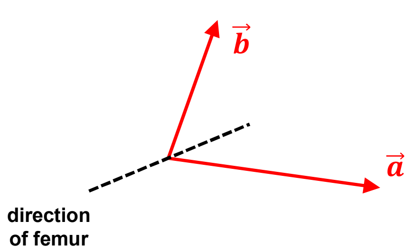

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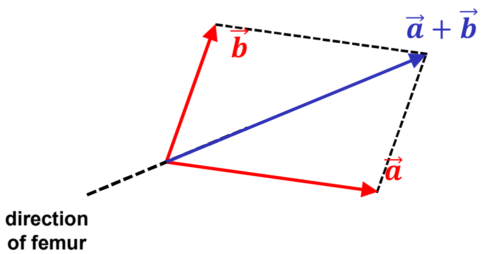
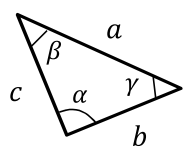


Pulleys and weights are arranged such that the resultant vector is aligned to the femur's direction

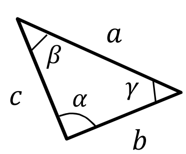
Let's recall…



Let's recall…

- Sine rule

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$



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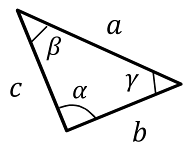


- Cosine rule

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

Let's recall…

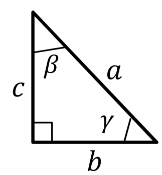
- Right triangles

$$a^2 = b^2 + c^2$$

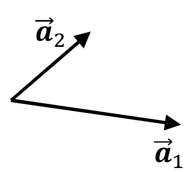
$$\sin\beta = \frac{opposite\ side}{hypotenuse}$$

$$\cos\beta = \frac{adjacent\ side}{hypotenuse}$$

$$\tan\beta = \frac{opposite\ side}{adjacent\ side}$$



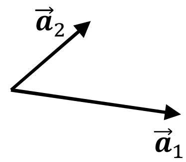
Vector cross product



Vector cross product

- It is a vector

$$\vec{\boldsymbol{V}} = \vec{\boldsymbol{a}}_1 \times \vec{\boldsymbol{a}}_2$$



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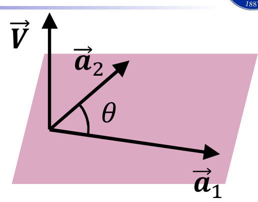


- $\vec{\boldsymbol{V}}$ is perpendicular to the plane of $\vec{\boldsymbol{a}}_1$ and $\vec{\boldsymbol{a}}_2$

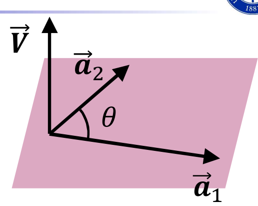
Vector cross product

- It is a vector

$$\vec{\boldsymbol{V}} = \vec{\boldsymbol{a}}_1 \times \vec{\boldsymbol{a}}_2$$

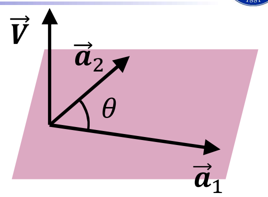
- The magnitude is:

$$V = a_1 a_2 \sin\theta$$



- $\vec{\boldsymbol{V}}$ is perpendicular to the plane of $\vec{\boldsymbol{a}}_1$ and $\vec{\boldsymbol{a}}_2$

Vector cross product



- It is a vector

$$\vec{V} = \vec{a}_1 \times \vec{a}_2$$

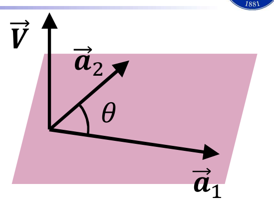
- The magnitude is:

$$V = a_1 a_2 \sin\theta$$

It is the smallest angle between $\vec{a}_1$ and $\vec{a}_2$

- $\vec{V}$ is perpendicular to the plane of $\vec{a}_1$ and $\vec{a}_2$

Vector cross product



- It is a vector

$$\vec{V} = \vec{a}_1 \times \vec{a}_2$$

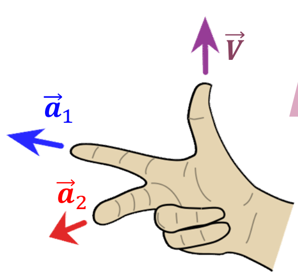
- The magnitude is:

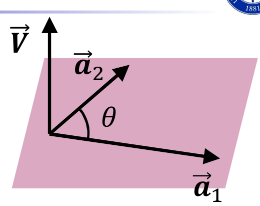
$$V = a_1 a_2 \sin\theta$$

It is the smallest angle between $\vec{a}_1$ and $\vec{a}_2$

- $\vec{V}$ is perpendicular to the plane of $\vec{a}_1$ and $\vec{a}_2$
- The direction of $\vec{V}$ is found by using the right-hand rule

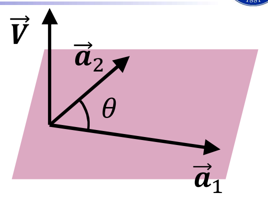
Right-hand rule





Vector cross product

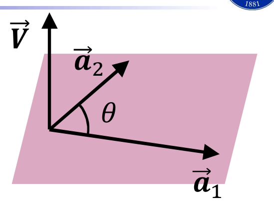
$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$$
$$\boldsymbol{j} \times \boldsymbol{i} = -\boldsymbol{k}$$



Vector cross product

$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$$
$$\boldsymbol{j} \times \boldsymbol{i} = -\boldsymbol{k}$$

$$\boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j}$$
$$\boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$$

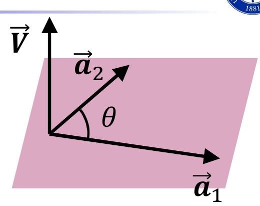


Vector cross product

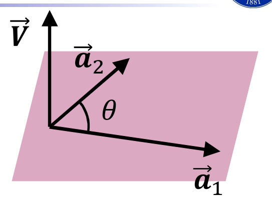
$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$$
$$\boldsymbol{j} \times \boldsymbol{i} = -\boldsymbol{k}$$

$$\boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j}$$
$$\boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$$

$$\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$$
$$\boldsymbol{k} \times \boldsymbol{j} = -\boldsymbol{i}$$



Vector cross product



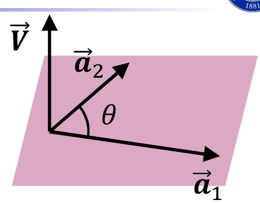
$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$$
$$\boldsymbol{j} \times \boldsymbol{i} = -\boldsymbol{k}$$

$$\boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j}$$
$$\boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$$

$$\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$$
$$\boldsymbol{k} \times \boldsymbol{j} = -\boldsymbol{i}$$

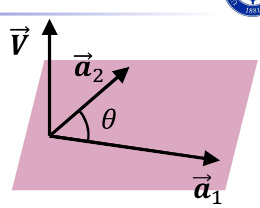
$$\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = \boldsymbol{0}$$

Vector cross product



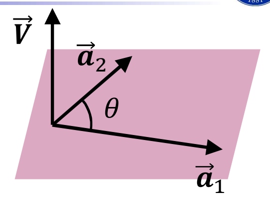
$$\vec{\boldsymbol{V}} = \vec{\boldsymbol{a}}_1 \times \vec{\boldsymbol{a}}_2$$

Vector cross product



$$\vec{\boldsymbol{V}} = \vec{\boldsymbol{a}}_1 \times \vec{\boldsymbol{a}}_2$$
$$= \left(a_{1,x}\boldsymbol{i} + a_{1,y}\boldsymbol{j} + a_{1,z}\boldsymbol{k}\right) \times \left(a_{2,x}\boldsymbol{i} + a_{2,y}\boldsymbol{j} + a_{2,z}\boldsymbol{k}\right)$$

Vector cross product



$$\vec{V} = \vec{a}_1 \times \vec{a}_2$$

$$= \left(a_{1,x}\boldsymbol{i} + a_{1,y}\boldsymbol{j} + a_{1,z}\boldsymbol{k}\right) \times \left(a_{2,x}\boldsymbol{i} + a_{2,y}\boldsymbol{j} + a_{2,z}\boldsymbol{k}\right)$$

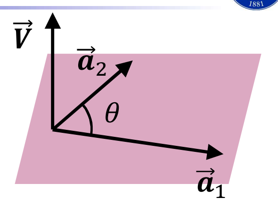
$$= \left(a_{1,y}a_{2,z} - a_{1,z}a_{2,y}\right)\boldsymbol{i}$$

$$+\left(a_{1,z}a_{2,x} - a_{1,x}a_{2,z}\right)\boldsymbol{j}$$

$$+\left(a_{1,x}a_{2,y} - a_{1,y}a_{2,x}\right)\boldsymbol{k}$$

Vector cross product

$$V_x = a_{1,y}a_{2,z} - a_{1,z}a_{2,y}$$
$$V_y = a_{1,z}a_{2,x} - a_{1,x}a_{2,z}$$
$$V_z = a_{1,x}a_{2,y} - a_{1,y}a_{2,x}$$



Vector cross product

$$V_x = a_{1,y}a_{2,z} - a_{1,z}a_{2,y}$$
$$V_y = a_{1,z}a_{2,x} - a_{1,x}a_{2,z}$$
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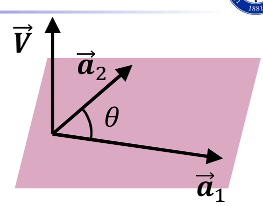

$$\vec{V} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_{1,x} & a_{1,y} & a_{1,z} \\ a_{2,x} & a_{2,y} & a_{2,z} \end{vmatrix}$$