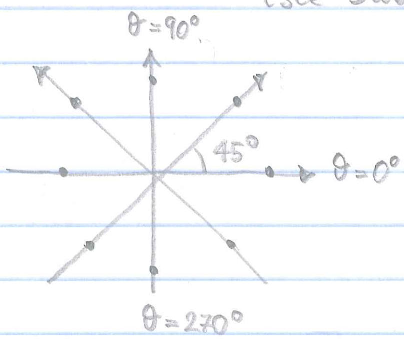
LECTURE 3

Having a model that relates measurements (input/output variables) to the variables to be estimated is the premise for using the KFs

If the measurements are neural recordings and the state variables are kinematic variables pertaining to a specific movement, what model do we have?

- Example 1: First attempt to relate firing rate (of MI neurons) to direction of movement by Georgopoulos et. al, 1982 (see slides)



$y_\theta \triangleq$ avg. firing rate during the movement of direction ϑ

$$y_\vartheta = b_0 + b_1 \sin\vartheta + b_2 \cos\vartheta \quad (1)$$

Estimation of $X \triangleq [b_0 \; b_1 \; b_2]^T$ can be obtained by using LS:

$$Y \triangleq \begin{bmatrix} y_0 \\ y_{45} \\ y_{90} \\ \vdots \\ y_{315} \end{bmatrix} \qquad A \triangleq \begin{bmatrix} 1 & 0 & 1 \\ 1 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 1 & 1 & 0 \\ & \cdots & \end{bmatrix}$$

8×1 $\qquad$ 8×3

← it is obtained by evaluating $\sin\vartheta$ and $\cos\vartheta$ for each value of $\vartheta \in \{0°; 45°; 90°; \ldots; 315°\}$

$$Y = AX \Rightarrow \hat{X} = (A^TA)^{-1}A^TY$$

It can be shown that, for the specific A assigned here:

$$\hat{b}_0 = \frac{1}{8}\sum_\vartheta y_\vartheta \quad \text{- sample mean}$$

(2)

$$\hat{b}_1 = \frac{1}{4}\left[\frac{1}{\sqrt{2}}\left(y_{45} + y_{135} - y_{225} - y_{315}\right) + \left(y_{90} - y_{270}\right)\right]$$

$$\hat{b}_2 = \frac{1}{4}\left[\frac{1}{\sqrt{2}}\left(y_{45} - y_{135} - y_{225} + y_{315}\right) + \left(y_{0} - y_{180}\right)\right]$$

The LS solution allows two considerations:

- We need a way to estimate the goodness-of-fit of the model on the data $\Rightarrow$ One can consider the fraction of variance in the data explained by the model:

"coefficient of determination" $\rightarrow$ $$R^2 \triangleq \frac{\underset{\theta}{E}\left[(\hat{y}_\theta - \bar{\hat{y}})^2\right]}{\underset{\theta}{E}\left[(y_\theta - \bar{y})^2\right]}$$

$\hat{y}_\theta \triangleq$ prediction given by (1)

$\bar{\hat{y}} \triangleq$ mean value of $\hat{y}_\theta$

$\bar{y} \triangleq$ mean value of measurements y_θ

Note: $$\bar{\hat{y}} \cong \frac{1}{8}\sum_\theta \hat{y}_\theta = \frac{1}{8}\sum_\theta \left(\hat{b}_0 + \hat{b}_1 \sin\theta + \hat{b}_2 \cos\theta\right) = \hat{b}_0$$

$\uparrow$ because of the symmetry of sin and cos functions

Therefore: $$\underset{\theta}{E}\left[(\hat{y}_\theta - \bar{\hat{y}})^2\right] =$$

$$\cong \frac{1}{8}\sum_\theta \left(\hat{b}_1 \sin\theta + \hat{b}_2 \cos\theta\right)^2 =$$

$$= \frac{1}{8}\sum_\theta \left(\hat{b}_1^2 \sin^2\theta + \hat{b}_2^2 \cos^2\theta + 2\hat{b}_1\hat{b}_2 \sin\theta\cos\theta\right)$$

$$= \frac{1}{8}\cdot 4\left(\hat{b}_1^2 + \hat{b}_2^2\right)$$

$\uparrow$ Because θ is evenly spaced between $0°$ and $315°$

$\Downarrow$

$$R^2 = \frac{4\left(\hat{b}_1^2 + \hat{b}_2^2\right)}{\sum_\theta (y_\theta - \hat{b}_0)^2}$$

- We can use model (1) to predict the preferred direction of movement $\Rightarrow$ Remember the following trigonometric identity:

$$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

We can set:

$$\left.\begin{array}{l} \theta \rightarrow \alpha \\ \theta_r \rightarrow \beta \text{ - preferred direction (to be chosen)} \end{array}\right\} \Rightarrow \begin{array}{l} b_1 \propto \sin(\theta_r) \\ b_2 \propto \cos(\theta_r) \end{array}$$

Therefore, an estimation of θ_r is:

$$b_1 \sin\theta + b_2 \cos\theta = \sqrt{b_1^2+b_2^2}\left[\frac{b_1}{\sqrt{b_1^2+b_2^2}}\sin\theta + \frac{b_2}{\sqrt{b_1^2+b_2^2}}\cos\theta\right] \Rightarrow$$

$$\hat{\theta}_r = tg^{-1}\left(\frac{\hat{b}_1}{\hat{b}_2}\right)$$ - Model (1) can be written in the form:

$$y_\theta = b_0 + c_1 \cos(\theta - \theta_r) \qquad (1')$$

i.e, we predict that the neuron will modulate its firing rate (compared to the baseline firing rate value) depending on the distance between actual direction and preferred direction $\Rightarrow$ Max increase in firing rate happens at the preferred direction:

$$I \triangleq \frac{\hat{c}_1}{\hat{b}_0}$$ - is an index of the directional modulation

A few considerations on this example:

- Does every neuron in M1 satisfy eq (1')? No, it does not $\rightarrow$ The study shows that only a fraction of cells (241 out of 606, i.e, ~40% of the recorded neurons) show this behavior, i.e., tuning

- Does every neuron encode a unique direction? No, it does not $\rightarrow$ The study shows that groups of neurons have preferred directions in the same 45°-wide cone and that, as a certain direction is taken, some neurons increase the firing rate and some decrease it ⇒ There is certainly redundancy

- Does every neuron encode a direction in the same way? No, it does not → The study shows that every neuron has different b_0, c_1, and I

- What does an encoding neuron really encode? We do not know. However we can likely rule out:
  - sensory feedback → The modulation of the firing rate precedes the EMG of the arm and the movement + It starts as early as 60-80 ms after the direction cue
  - direction (i.e., angle) → The neuron does not know about direction. It likely contributes to encode the sequence of motor commands to be projected down to the muscles.

The activity is related to the "movement intent"

"Motor commands" is willingly vague at this point and, depending on the specific definition that is given (e.g., force, error signal, etc.), different approaches have been developed.

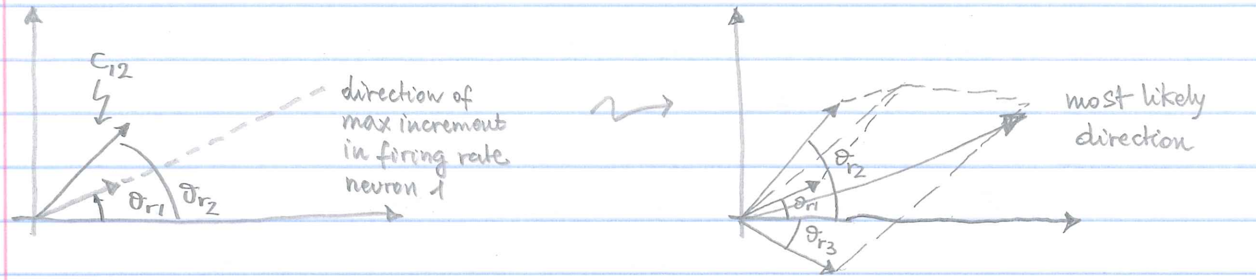
One Implication of this study is: most of the cells that increase (or decrease) the firing rate around movement onset are tuned with the direction of the movement ⇒ If enough cells are given, the sum of the vector angles can predict the direction of movement:

neuron 1: $y_1 = b_{01} + c_{11} \cos(\vartheta - \vartheta_{r1})$

neuron 2: $y_2 = b_{02} + c_{12} \cos(\vartheta - \vartheta_{r2})$

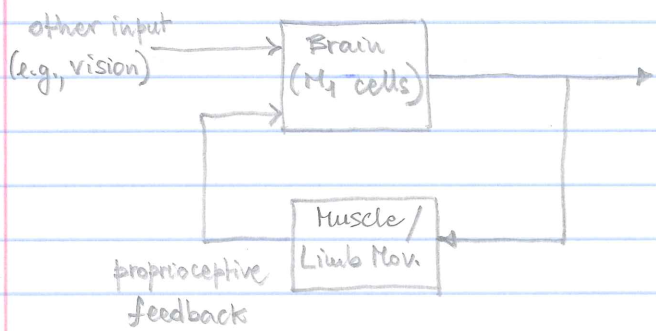
$\vdots$

neuron N: $y_N = b_{0N} + c_{1N} \cos(\vartheta - \vartheta_{rN})$



- Example 2: If a population of neurons in M_1 can collectively encode the movement intent, then two questions arise:
  - can the population activity be used to retrieve the goal position?
  - is the encoding affected by proprioceptive feedback from muscle activation and limb movement?

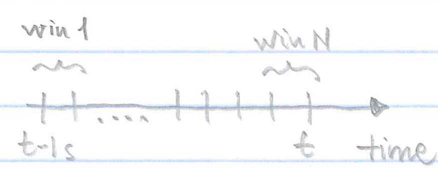
These two questions were answered by Serruya et al., 2002 (see slides)



Given the random movement of the goal position, processing each neuron's firing rate as an independent predictor of the goal position may not be enough ⟹ A multivariate approach is needed:

$$x_t = \sum_{j=1}^{N} \left(a_{1j}\, n_{1j_t} + a_{2j}\, n_{2j_t} + \dots + a_{Nj}\, n_{Nj_t} \right)$$

$$y_t = \sum_{j=1}^{N} \left(b_{1j}\, n_{1j_t} + b_{2j}\, n_{2j_t} + \dots + b_{Nj}\, n_{Nj_t} \right)$$



win size = 50 ms

$x_t, y_t \triangleq$ position at time t along the x and y direction, respectively

$n_{ij_t} \triangleq$ f.r. of neuron j in win #i before time t

By combining the firing rates of all neurons in a matrix, we obtain a standard LS problem:

$$X \triangleq \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \cdots & \\ x_m & y_m \end{bmatrix}$$

$m \times 2$ matrix, with m being the number of goal positions that have been considered

$$Y \triangleq \underbrace{\begin{bmatrix} n_{11_{t=1}} & n_{21_{t=1}} & \cdots & n_{N1_{t=1}} & n_{12_{t=1}} & n_{22_{t=1}} & \cdots & n_{N2_{t=1}} & \cdots & n_{NK_{t=1}} \\ & & & & \vdots & & & & & \\ n_{11_{t=m}} & n_{21_{t=m}} & \cdots & n_{N1_{t=m}} & & \cdots & & & & n_{NK_{t=m}} \end{bmatrix}}_{m \times (Nk) \text{ neural response matrix}}$$

← $N=20$ is the # of windows for each of k neurons

$$F = \begin{bmatrix} a_{11} & b_{11} \\ a_{21} & b_{21} \\ \vdots & \vdots \\ a_{N1} & b_{N1} \\ a_{12} & b_{12} \\ \vdots & \vdots \\ a_{NK} & b_{NK} \end{bmatrix}$$

$(Nk) \times 2$ matrix of coefficients

By using the matrix notation:

$YF = X \Rightarrow$ In order to estimate F from measurements Y and X, we can use the formula:

$$\hat{F} = (Y^T Y)^{-1} Y^T X$$

This matrix $\hat{F}$ can then be used to estimate the kinematic variable $\hat{X}_t$ at any new time t by using the newly acquired measurements Y_t at time t and in the previous second:

$$\hat{X}_t = Y_t \hat{F}$$

The study poses a few questions:

1) How many neurons do we need?
2) What kind of neurons do we need?
3) How to assess the performance of the decoder?

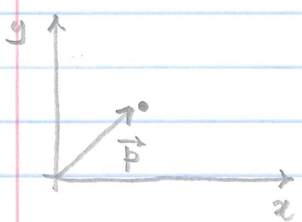
Questions 1) and 2) are deeply related to the framework (i.e., the fact that we use a specific-linear-approach) and the physiology of the brain $\Rightarrow$ They must answered by looking at the data

$\Downarrow$

- Example 3: Paninski et al., 2004, focused on these two questions and showed:

    a) As low as 12 neurons in M_1 may suffice
    b) Neurons do NOT need to be "cherry-picked"

While the number of neurons (12) may depend on the experiment, the fact that any population of M_1 neurons may be used to estimate kinematic variables depends on the receptive field of each neuron:



$$N(\vec{p}, \tau) \triangleq \mathop{E}_{t}\left[n_t \,\middle|\, \vec{p}(t+\tau) = \vec{p} \right]$$

conditional probability of having firing rate n_t at time t, conditioned to being in position $\vec{p}$ at time $t+\tau$

Similarly, one can calculate: $N(\vec{v}, \tau) \triangleq \mathop{E}_{t}\left[n_t \,\middle|\, \vec{v}(t+\tau) = \vec{v} \right]$ by using the velocity vector $\vec{v}$ $\Rightarrow$ It can be shown from recordings that neurons have a broad-shaped field for both position and velocity (see slides)

Interestingly enough, if $\vec{v}$ is expressed in polar coordinates (ρ, θ) ($\rho \triangleq$ radial speed; $\theta \triangleq$ direction), a cosine-based model can describe the tuning curve:

$$N(\vec{v}, \tau) = N(\rho, \theta, \tau) \cong a_0 + a_1(\tau) \cos(\theta - \theta_r)$$

with $\theta_r \triangleq$ preferred direction

The determination of the number of neurons to be used for decoding is a consequence of the way question 3) was answered by the authors

$\Downarrow$

Instead of a deterministic measure of accuracy (e.g., distance between actual and predicted kinematics), they made the following considerations:

$$S(t) \triangleq \begin{bmatrix} \vec{p} \\ \vec{v} \end{bmatrix}$$ - vector (4×1) of position and velocity at time t

$$N(t) \triangleq \begin{bmatrix} n_{1t} \\ n_{2t} \\ \vdots \\ n_{kt} \end{bmatrix}$$ - vector $(k \times 1)$ of firing rates at time t (one per neuron)

If - as suggested by the receptive field study - every neurons may modulate its activity several ms before a certain position/velocity is reached, then a prediction of S from N does have to include many observation bins:

$$S(t) \longleftrightarrow \mathbb{N}_t \triangleq \begin{bmatrix} N(t) \\ N(t-1) \\ \vdots \\ N(t-\tau) \end{bmatrix}$$ $\Rightarrow$ The estimation problem is over-determined

Therefore the criteria chosen to assess the performance of the decoder consist in:

- Measuring the amount of variance in S that is explained by $[N(t)\ N(t-1) \ldots]^T$
- Truncating the vector of predictors when no gain in explained variance is obtained
- Testing how sensitive this procedure is to the specific selection of k neurons

A measure of the explained variance is:

$$r^2 = 1 - \frac{\underset{t}{E}\left[\left(S(t) - N_t \right)^2 \right]}{\underset{t}{E}\left[S(t)^2 \right]}$$

(assumed that $S(\cdot)$ has been mean-removed) $\Rightarrow$ See slides for results. □

Decoding performance is certainly an issue for the formulations considered thus far $\Rightarrow$ Two contributions to that are:

- neurons have a wide range of kinematic values they relate to
- there is interdependency and redundancy in the representation of the kinematic variables

$\Downarrow$

A paradigm shift can be provided by Bayesian Estimation. Let us consider the following example:

$x_t \triangleq$ position at time t — $x \triangleq$ assigned position

$n_t \triangleq$ firing rate of a given neuron at time t — $n \triangleq$ assigned firing rate

$$\left.\begin{aligned} Pr(x_t = x, n_t = n) &= Pr(x_t = x \mid n_t = n)\, Pr(n_t = n) \\ &= Pr(n_t = n \mid x_t = x)\, Pr(x_t = x) \end{aligned}\right\}$$ From the definition of conditional probability

$\downarrow$

For the probability distribution functions:

$$\underset{x}{p}(x_t \mid n_t) = \underset{n}{p}(n_t \mid x_t) \frac{p_x(x_t)}{\underset{n}{p}(n_t)} \qquad (*)$$

$p_n \triangleq$ pdf of firing rate

$p_x \triangleq$ pdf of position

Note: $p_n(n_t \mid x_t) \longleftrightarrow N(\vec{p}, 0)$ defined before and can be estimated from data $\Rightarrow$ It says how likely the firing rate n_t is assuming x_t known

10

$p_x(x_t) \longleftrightarrow$ It describes the probability of being in any point in space regardless any neural observation

$p_n(n_t) \longleftrightarrow$ It describes the probability of making an observation and, in equation (*), it acts as a normalization factor (to ensure that the integration sums to 1)

In our case: n_t is measured from actual observations $\Rightarrow$ $p_n(n_t)$ is called "marginal" pdf

x_t has to be estimated $\Rightarrow$ $p_x(x_t)$ is called "prior" of x_t

$p_x(x_t | n_t)$ is called the "posterior" pdf

Let us assume, as in the presentation of the KF, that we have a model describing x_t $\Rightarrow$ We can build $p_x(x_t)$ from the model and we can think of $p_n(n_t|x_t)$ as the pdf of n_t if the model of x_t were true

Now, in order to fully exploit the Bayesian framework, we need to make assumptions about the model of x_t and the relationship between x_t and n_t:

A1) $n_t = A x_t + \varepsilon_t$ (as we did before) $\quad \varepsilon_t \sim N(0, R_t)$

$$\underset{n}{E}(n_t) = A \underset{x}{E}(x_t)$$

$\Rightarrow$ Therefore, if we assume that a prior estimate $\hat{x}_t^-$ is given and it has a probability distribution $N(\hat{x}_t^-, \hat{P}_t^-)$, we have:

$$\underset{n}{E}(n_t) = A \hat{x}_t^-$$

$$var(n_t) = \underset{n}{E}\left[(n_t - A\hat{x}_t^-)^2\right] = A \hat{P}_t^- A^T + R$$

Therefore, through assumptions A1), we are saying that the firing rate is expected to be normally distributed according to the function

$$N\left(A\hat{x}_t^-,\ A\hat{P}_t^- A^T + R\right)$$

This can be used to ultimately determine the posterior pdf. In fact:

$$p_x(x_t \mid n_t) = \frac{p(x_t, n_t)}{p_n(n_t)}$$

From above, we know $p_n(n_t) \sim N\left(A\hat{x}_t^-,\ A\hat{P}_t^- A^T + R\right)$

We need to estimate $p(x_t, n_t) \Rightarrow$ We need $\mathrm{cov}(x_t, n_t)$

$$\mathrm{cov}(x_t, n_t) = E\left[(x_t - \hat{x}_t^-)(Ax_t + \varepsilon_t - A\hat{x}_t^-)\right] = A\hat{P}_t^-$$

Now: $\left.\begin{array}{l}\hat{x}_t^- \text{ is a Normal pdf} \\ n_t \text{ is a Normal pdf}\end{array}\right\} \Rightarrow p(x_t, n_t) \sim N\left(\begin{bmatrix}\hat{x}_t^- \\ A\hat{x}_t^-\end{bmatrix}, \begin{bmatrix}\hat{P}_t^- & \hat{P}_t^T A^T \\ A\hat{P}_t^- & A\hat{P}_t^- A^T + R\end{bmatrix}\right)$

From calculation (we will do it, in part, next lecture), it can be shown that – because $p_n(n_t)$ and $p(x_t, n_t)$ are both normal – the posterior probability $p(x_t \mid n_t)$ is normal with mean and variance:

$$\mathrm{var}(x_t \mid n_t) = \left[(\hat{P}_t^-)^{-1} + A^T R^{-1} A\right]^{-1}$$

$$\underset{x}{E}(x_t \mid n_t) = \hat{x}_t^- + \left[(\hat{P}_t^-)^{-1} + A^T R^{-1} A\right] A^T R^{-1} (n_t - A\hat{x}_t^-)$$

$$\Downarrow$$

We obtain the equations of the KF (correction part) $\Rightarrow$ The KF can be used to estimate mean and variance of the posterior probability of the hidden state (i.e., position) to be estimated

□